A comparative assessment of measures of similarity of fuzzy values

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Abstract: The properties of several measures of similarity of fuzzy values are presented and compared. The measures examined include the measure based on the union and intersection, the one based on the maximum difference and the one based on the differences as well as the sum of corresponding grades of membership. It is shown that several properties are common to all measures. However, some properties do not hold for all of them.

Keywords: Fuzzy value; measure of similarity.

1. Introduction

Approximation is inherent in fuzzy set theory. It has been noted that small deviations from what might be considered as “precise membership values” should normally be considered of no practical significance [4]. On the other hand, approximation is implied when considering, for example, the multitude of solutions of the Inverse Problem (i.e., “given a fuzzy relation R from U to V and a fuzzy subset B of V, find all fuzzy subsets A of U such that A o R = B, where o denotes maximin composition”) [3].

The notion of approximation of fuzzy values has been discussed and the definitions of proximity measure and approximately equal fuzzy sets have been introduced in [1]. A measure of similarity, based on the maximum difference of corresponding grades of membership, was implied and several properties of fuzzy sets in relation to this measure were shown.

An alternative measure of similarity of fuzzy values, namely the grade of similarity, based on the notion of proximity measure, was introduced in [2]. It was shown that this grade of similarity, based on the differences as well as the sum of corresponding grades of membership, has several interesting properties.

In this paper, one more such measure, based on the union and intersection of corresponding grades of membership, is defined, its properties are shown and the properties of all three measures are summarized and compared. It is shown that, while several properties are common to these measures, there are remarkable differences between them, which may influence the final choice of the measure to be used in applications of fuzzy sets.

2. Notations and definitions

Let a and b be scalars. The following notations will be used:

\( a \wedge b \): \( \min(a, b) \),

\( a \vee b \): \( \max(a, b) \).

Also \( A^c \) denotes the complement of A, while \( I, O \) and \( M \) denote the unit, zero and 0.5 fuzzy sets, respectively, i.e., those with all grades of membership equal to 1, 0 or 0.5, respectively.

The \( \circ \) composition of the vector \( a = (a_1, a_2, \ldots, a_m) \), corresponding to the fuzzy subset \( A \) of \( U = \{u_i \mid i = 1, \ldots, m\} \), with the matrix \( R = [r_{ij}] \), corresponding to the fuzzy relation \( R \) of \( U \times V \), is denoted by \( a \circ R \) and is equal to the vector \( c = (c_1, c_2, \ldots, c_n) \) where

\[ c_j = \vee (a_i \wedge r_{ij}). \]

The \( \alpha \) composition of a scalar \( a \) with a scalar \( b \), denoted by \( a \alpha b \), is defined as follows:

\[ a \alpha b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{otherwise}. \end{cases} \]

The \( \alpha \) composition of the vector
\( x = (x_1, x_2, \ldots, x_n) \) with the scalar \( a \) is formed by substituting each element \( x_i \) of \( x \) with \( x_i \alpha a \).

The \( \alpha \) composition of the matrix \( R \) with the vector \( x = (x_1, x_2, \ldots, x_n) \) denoted by \( R \alpha x \), is formed by substituting each column vector \( r_i \) of \( R \) with \( r_i \alpha x_i \). \( \wedge (R \alpha x) \) denotes the vector whose elements are formed by taking the minimum element of the respective row vector of \( R \alpha x \).

3. Measure based on the operations of union and intersection

The grade of similarity \( M_{A,B} \) of the fuzzy sets \( A \) and \( B \), is defined by

\[
M_{A,B} = \frac{\sum_i (a_i \vee b_i)}{\sum_i (a_i \wedge b_i)}.
\]

\( A \) and \( B \) are said to be approximately equal (denoted by \( A \sim B \)) iff, given a small nonnegative number \( \varepsilon \), it is

\[
M_{A,B} \leq \varepsilon.
\]

The number \( \varepsilon \) is said to be a proximity measure of \( A \) and \( B \).

Properties of \( M_{A,B} \). The following properties of \( M_{A,B} \) are true:

(M1) \( M_{A,B} = M_{B,A} \).
(M2) \( A = B \iff M_{A,B} = 1 \).
(M3) \( A \wedge B = 0 \iff M_{A,B} = 0 \).
(M4) \( M_{A,A} = 1 \iff A = M \).
(M5) \( M_{A,A} = 0 \iff A = I \) or \( A = O \).

Properties of approximately equal fuzzy sets.

(M6) \( A \sim B \) does not necessarily imply that \( A \vee C \sim B \vee C \).

Consider \( a = (0.4 \ 0.8 \ 1.0), \ b = (0.5 \ 0.7 \ 0.8), \) and \( c = (0.4 \ 0.6 \ 0.9) \). It follows that

\[
M_{A,B} = \frac{0.4 + 0.7 + 0.8}{0.5 + 0.8 + 1.0} = \frac{1.9}{2.3} = 0.826
\]

and

\[
M_{A \vee C,B \vee C} = \frac{0.4 + 0.7 + 0.9}{0.5 + 0.8 + 1.0} = \frac{2.0}{2.3} = 0.869,
\]

i.e., \( M_{A \vee C,B \vee C} > M_{B,A} \), which means that the proximity measure of \( A \vee C \) and \( B \vee C \) is greater than that of \( A \) and \( B \).

Similarly, the following can be shown:

(M7) \( A \sim B \) does not necessarily imply that \( A \wedge C \sim B \wedge C \).

Maximin composition.

(M8) \( A \sim B \) does not necessarily imply that \( A \circ R \sim B \circ R \).

Consider \( a = (0.4 \ 0.6 \ 0.6), \ b = (0.5 \ 0.7 \ 0.8), \)

\[
R = \begin{bmatrix}
0.2 & 1.0 \\
0.6 & 0.6 \\
1.0 & 0.3
\end{bmatrix}
\]

and let \( A \circ R = C, \ B \circ R = D. \) It follows that \( c = (0.6 \ 0.6), \ d = (0.8 \ 0.6) \)

\[
M_{A,B} = \frac{0.4 + 0.6 + 0.6}{0.5 + 0.7 + 0.8} = \frac{1.6}{2.0} = 0.8
\]

and

\[
M_{C,D} = \frac{0.6 + 0.6}{0.8 + 0.6} = \frac{1.2}{1.4} = 0.857,
\]

i.e., \( M_{C,D} > M_{A,B} \), which means that the proximity measure of \( A \circ R \) and \( B \circ R \) is greater than that of \( A \) and \( B \).

Similarly, it can be shown that, if \( S \) is a fuzzy relation from \( U \) to \( V \), then

(M9) \( R \sim S \) does not necessarily imply that \( A \circ R \sim A \circ S \).

\( \alpha \)-composition. Let \( \wedge (R \alpha x) = f \) and \( \wedge (R \alpha y) = g \), \( F, G \) be the fuzzy sets with membership vectors equal to \( f \) and \( g \), respectively.

(M10) \( X \sim Y \) does not necessarily imply that \( F \sim G \).

Consider

\[
R = \begin{bmatrix}
1.0 & 0.9 & 0.8 \\
0.6 & 0.9 & 0.5 \\
0.5 & 1.0 & 0.2
\end{bmatrix},
\]

\[
x = (0.6 \ 0.8 \ 0.5) \) and \( y = (0.5 \ 0.9 \ 0.8) \). It follows that \( f = \wedge (R \alpha x) = (0.50.80.8), \ g = \wedge (R \alpha y) = (0.5 \ 0.5 \ 0.9) \)

\[
M_{X,Y} = \frac{0.5 + 0.8 + 0.6}{0.6 + 0.9 + 0.8} = \frac{1.8}{2.3} = 0.782
\]

and

\[
M_{F,G} = \frac{0.5 + 0.5 + 0.8}{0.5 + 0.8 + 0.9} = \frac{1.8}{2.2} = 0.818.
\]
Thus, $M_{F,G} > M_{X,Y}$, i.e., the grade of similarity of $F$ and $G$ is greater than that of $X$ and $Y$, thus $X \sim Y$ does not necessarily imply that $F \sim G$.

Similarly, it can be shown that if $R$ and $S$ are the matrices corresponding to the fuzzy relations $R$ and $S$, if $\bigwedge (R \alpha x) = f$ and $\bigwedge (S \alpha x) = k$, then

$$(M11) \ R \sim S \text{ does not necessarily imply that } F \sim K.$$  

Note that $\bigwedge (R \alpha x) = f$ is the upper bound of the solution (if it exists) of the inverse problem “given $R$ and $x$ find $f$ such that $f \circ R = x$”.

4. Measure based on the maximum difference

The grade of similarity $L_{A,B}$ of the fuzzy sets $A$ and $B$, is defined by

$$L_{A,B} = 1 - \max_i |a_i - b_i|$$

The definitions of approximately equal fuzzy sets and proximity measure in the case of $L_{A,B}$ are similar to those of Section 3.

The following properties of $L_{A,B}$ are true:

(L1) $L_{A,B} = L_{B,A}$.
(L2) $A = B \iff L_{A,B} = 1$.
(L3) $A \wedge B = 0 \iff L_{A,B} = 1 - \max_i (a_i, b_i)$.
(L4) $L_{A,A} = 1 \iff A = M$.
(L5) $L_{A,A} = 0 \iff A$ or $A^c$ are normal fuzzy sets.
(L6) $A \sim B \Rightarrow A \vee C \sim B \vee C$.
(L7) $A \sim B \Rightarrow A \wedge C \sim B \wedge C$.
(L8) $A \sim B \Rightarrow A \circ R \sim B \circ R$.
(L9) $R \sim S \Rightarrow A \circ R \sim A \circ S$.
(L10) $X \sim Y$ does not necessarily imply that $F \sim G$.
(L11) $R \sim S$ does not necessarily imply that $F \sim K$.

5. Measure based on the difference and the sum of grades of membership

The grade of similarity $S_{A,B}$ of $A$ and $B$, is defined by

$$S_{A,B} = 1 - \frac{\sum_i |a_i - b_i|}{\sum_i (a_i + b_i)}$$

or, equivalently, by

$$S_{A,B} = 1 - \frac{\sum_i (a_i \vee b_i - a_i \wedge b_i)}{\sum_i (a_i + b_i)}.$$

The definitions of approximately equal fuzzy sets and proximity measure in the case of $S_{A,B}$ are similar to those of Section 3.

The following properties of $S_{A,B}$ are true [2]:

(S1) $S_{A,B} = S_{B,A}$.
(S2) $A = B \iff S_{A,B} = 1$.
(S3) $A \wedge B = 0 \iff S_{A,B} = 0$.
(S4) $S_{A,A} = 1 \iff A = M$.
(S5) $S_{A,A} = 0 \iff A = I$ or $A = O$.
(S6) $A \sim B \Rightarrow A \vee C \sim B \vee C$.
(S7) $A \sim B$ does not necessarily imply that $A \wedge C \sim B \wedge C$.
(S8) $A \sim B$ does not necessarily imply that $A \circ R \sim B \circ R$.
(S9) $R \sim S$ does not necessarily imply that $A \circ R \sim A \circ S$.
(S10) $X \sim Y$ does not necessarily imply that $F \sim G$.
(S11) $R \sim S$ does not necessarily imply that $F \sim K$.

6. A comparison of properties

Table 1 summarizes the properties of the three measures of similarity of fuzzy values discussed above. It can be seen that several of these properties are common to all measures. However, considerable differences between them do exist.

<table>
<thead>
<tr>
<th>Property</th>
<th>M</th>
<th>L</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $X_{A,B} = X_{R,A}$</td>
<td>Y*</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>2. $A = B \iff X_{A,B} = 1$</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>3. $A \wedge B = 0 \iff X_{A,B} = 0$</td>
<td>Y</td>
<td>N**</td>
<td>Y</td>
</tr>
<tr>
<td>4. $X_{A,A} = 1 \iff A = M$</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>5. $X_{A,A} = 0 \iff A = I$ or $A = O$</td>
<td>Y</td>
<td>N***</td>
<td>Y</td>
</tr>
<tr>
<td>6. $A \sim B \Rightarrow A \vee C \sim B \vee C$</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>7. $A \sim B \Rightarrow A \wedge C \sim B \wedge C$</td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>8. $A \sim B \Rightarrow A \circ R \sim B \circ R$</td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>9. $R \sim S \Rightarrow A \circ R \sim A \circ S$</td>
<td>N</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>10. $X \sim Y \Rightarrow \bigwedge (R \alpha x) \sim \bigwedge (R \alpha y)$</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>11. $R \sim S \Rightarrow \bigwedge (R \alpha x) \sim \bigwedge (S \alpha x)$</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

* Y = Yes, N = No.
** $I_{A,B} = 1 - \max_i (a_i, b_i)$.
*** $A$ or $A^c$ are normal fuzzy sets.
7. Conclusion

The definitions and the properties of three measures of similarity of fuzzy values, one of which has been introduced in this paper, have been presented and compared. It has been shown that several of these properties are common to all measures. However, several differences characterize them in respect to other properties. These differences should be kept in mind when selecting the appropriate measure in applications of fuzzy set theory.

References