Presliding Friction Identification Based Upon the Maxwell Slip Model Structure

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Abstract

The problem of presliding friction identification based upon the Maxwell Slip model structure, which is capable of accounting for the presliding hysteresis with nonlocal memory, is considered. The model structure’s basic properties are examined, based upon which a priori identifiability is established, the role of initial conditions on identification is investigated, and the necessary and sufficient conditions for a posteriori identifiability are derived. Using them, guidelines for excitation signal design are also formulated.

Building upon these results, two new methods, referred to as Dynamic Linear Regression (DLR) and NonLinear Regression (NLR), are postulated for presliding friction identification. Both may be thought of as different extensions of the conventional Linear Regression (LR) method that uses threshold preassignment: The DLR by introducing extra dynamics in the form of a vector finite impulse response filter, and the NLR by relaxing threshold preassignment through a special nonlinear regression procedure. The effectiveness of both methods is assessed via Monte Carlo experiments and identification based upon laboratory signals. The results indicate that both methods achieve significant improvements over the LR. The DLR offers the highest accuracy, with the NLR striking a very good balance between accuracy and parametric complexity.
Presliding friction is the ascendant friction phenomenon that dominates the behavior of positioning systems at velocity reversals or near the goal position. Its precise identification is thus important for purposes such as dynamic analysis, automatic control, and fault diagnosis. This study examines the problem of presliding friction identification based upon the Maxwell Slip model structure, which is capable of accounting for the presliding hysteresis with nonlocal memory phenomena present. The model structure’s necessary and sufficient conditions for identifiability are derived, and the role of initial conditions on identification is investigated. Two new identification methods are then postulated: A Dynamic Linear Regression (DLR) method and a Nonlinear Regression (NLR) method. Both may be thought of as different extensions of the conventional Linear Regression (LR) method that uses threshold preassignment. The effectiveness of the methods is assessed via Monte Carlo experiments and identification based upon laboratory signals. The results indicate that both methods achieve significant improvements over the LR. The DLR offers the highest accuracy, with the NLR striking a very good balance between accuracy and parametric complexity.
1 Introduction

Friction is the result of an extremely complex interaction between two surfaces, as well as other substances (such as lubricants) that may be present. It is well known that friction includes two operating regimes, referred to as the presliding regime and sliding regime [1].

In the presliding (micro-slip) regime, the relative displacement between the two contacting surfaces is infinitesimal (on the order of $2 \sim 5 \mu m$ for steel materials [1]). The friction force in this regime is mainly a function of displacement [1], and is due to the adhesive forces derived from the asperity junction elastoplastic deformation. As the displacement increases, more and more junctions break, and finally there is a breakaway displacement beyond which gross sliding (the sliding regime) begins. In the sliding regime the asperity junctions have been broken, and a macroscopic relative displacement of the surfaces in contact takes place. In this case the friction force is mainly a function of the relative velocity [1].

In many pointing and tracking applications with high precision requirements, presliding friction is the ascendant friction phenomenon that dominates the behavior of positioning systems at velocity reversals, or near the goal position. For this reason, precise presliding friction modelling is important for purposes such as dynamic analysis, fault diagnosis, and control.

Dedicated experiments focusing on the presliding friction regime [2, 3], have indicated that there is a hysteretic relationship characterized by nonlocal memory between the presliding displacement and the applied force. Such a relationship is characterized by dependency of the force not only on the current value of the displacement, but also on past extremum values of it [4].

Accurate modelling of presliding friction dynamics based upon first principles and material / surface properties is not possible to date. For this reason identification techniques based upon experimentally obtained data are typically used. These may be classified under two broad categories: Black-box methods, for which no prior knowledge on the presliding friction dynamics is utilized [5, 6], or grey-box methods (also referred to as physics based), for which some a priori information is used. Depending upon the form of the model structure, they may be also classified as state-independent and state-dependent.

The state-independent methods directly relate the presliding friction to the measured displacement. Kim et al. [7] employ the Armstrong model [1] (a linear spring connecting presliding friction and displacement) and attempt estimation based upon an evolutionary algorithm based optimization. Wu and
Tung [8] modify the Coulomb friction model by assuming that the presliding displacement is proportional to friction force changes, rather than the friction itself. Awabdy et al. [9] relate friction and displacement via the Todd and Johnson modified Dahl model [10] and estimate the parameters via curve fitting. In this case the presliding hysteresis seems to be described well, but parameter dependency upon the excitation is reported. It should be also noted that all of the aforementioned methods employ properly designed (dedicated) experiments.

The state-dependent methods relate the presliding friction to displacement via unmeasured state variables which account for the asperity deformations. The underlying dynamics is generally better described, but, at the same time, the identification becomes more challenging. A class of state-dependent methods is based upon the LuGre model structure [11] (and also its extension, referred to as the Elastoplastic friction model [12]) and suitable time domain identification using nonlinear optimization. Although their implementation is relatively simple, a sequence of properly designed (dedicated) experiments is, again, required. In an attempt to overcome this difficulty, a method based upon a linearized version of the LuGre model and frequency domain identification has been also proposed [13, 14]. Yet, none of these methods accounts for the hysteresis phenomena with nonlocal memory that are present in presliding friction.

An alternative method is based upon the Leuven friction model [3], in which two stacks that store the positive and negative displacement extrema associated with nested hysteresis loops, are utilized. Identification is based upon curve fitting, much like in the Preisach model case [4]. The advantage of this method is that hysteresis with nonlocal memory is accounted for; yet the size of the stacks must be preselected and in real-time systems this may result into stack overflow in cases where the number of the initiated loops exceeds the preselected value.

A recent attempt for overcoming this is based upon the Maxwell Slip model structure which utilizes a piecewise approximation of the presliding hysteresis through the use of a number of elementary operators subjected to Coulomb friction [15, 16]. Identification is based upon regression type criteria combined with the arbitrary prior assignment of certain physical parameters (thresholds).

This study aims at a detailed consideration of presliding friction identification based upon the Maxwell Slip model structure. This structure has the advantage of conceptual simplicity, coupled with the capability of accounting for the hysteresis with nonlocal memory phenomena. The model structure’s ba-
sic properties are examined, based upon which a priori identifiability is established, the role of initial conditions on identification is investigated, and the necessary and sufficient conditions for a posteriori identifiability are derived. Using these, guidelines for excitation signal design are also formulated.

Based upon these, two new methods, referred to as Dynamic Linear Regression (DLR) and NonLinear Regression (NLR), for presliding friction identification are postulated (also see [17]). Both may be thought of as different extensions of the conventional Linear Regression (LR) that uses threshold preassignment [15, 16]: The DLR by introducing extra dynamics in the form of a vector finite impulse response filter, and the NLR by relaxing threshold preassignment through a special nonlinear regression procedure. These methods also circumvent the need for dedicated experiments, as a single experiment under “usual” operating conditions suffices for proper identification. The effectiveness of both methods is assessed via Monte Carlo experiments and identification based upon laboratory signals, while comparisons with the LR method are also made.

The rest of this paper is organized as follows: The Maxwell Slip model structure is presented in section 2, in which its basic properties are also studied. The structure’s a priori and a posteriori identifiability are studied in section 3, where guidelines for excitation signal design are also formulated. The DLR and NLR identification methods are introduced in section 4, and their effectiveness is assessed via Monte Carlo experiments and actual laboratory signals in section 5. The conclusions from this study are finally summarized in section 6.

2 The Maxwell Slip Model Structure

2.1 Mathematical Description

The basic Maxwell Slip model structure is pictorially presented in Figure 1. It consists of a parallel configuration of \( M \) elasto-slide operators, each one characterized by negligible inertia. Each operator is characterized by its own linear stiffness \( k_i \) and maximum spring deformation \( \Delta_i \) (threshold). For spring deformations smaller, in magnitude, than \( \Delta_i \) the operator sticks; otherwise it slips. The operator’s current mass position is designated as \( x_i(t) \). The difference between the exerted (common to all operators) displacement \( x(t) \) and the \( i \)-th operator position \( x_i(t) \) provides the corresponding current spring
deformation $\delta_i(t)$:

$$\delta_i(t) \triangleq x(t) - x_i(t), \quad \text{with} \quad |\delta_i(t)| \leq \Delta_i, \quad \forall \ i = 1, \ldots, M \quad (1)$$

Obviously, $\delta_i(t) > 0$ corresponds to extension and $\delta_i(t) < 0$ to compression.

When the $i$-th operator sticks its mass is fixed and $\delta_i(t)$ changes proportionally to $x(t)$, with $|\delta_i(t)| < \Delta_i$. Alternatively, when it slips its mass is moving and $|\delta_i(t)|$ attains its maximum value ($|\delta_i(t)| = \Delta_i$).

Thus:

$$\frac{d\delta_i(t)}{dt} = \begin{cases} \frac{dx(t)}{dt} & \text{(Stick region)} \\ 0 & \text{(Slip region)} \end{cases} \quad (2)$$

The $i$-th operator remains sticking as long as $|\delta_i(t)| < \Delta_i$, while slipping until the exerted displacement reaches a local extremum, that is the exerted velocity $\left(v(t) = \frac{dx(t)}{dt}\right)$ goes through zero (changes sign).

The application of a forward differencing scheme on Eq.(2) yields (note that from this point on $t = 1, 2, \ldots$ refers to discrete time):

$$\delta_i(t + 1) = \begin{cases} \text{sgn}[x(t + 1) - x(t) + \delta_i(t)] \cdot |x(t + 1) - x(t) + \delta_i(t)|, & \text{(Stick regime)} \\ \text{sgn}[\delta_i(t)] \cdot \Delta_i, & \text{(Slip regime)} \end{cases} \quad (3)$$

Since $|\delta_i(t)| \leq \Delta_i, \forall \ t$, Eqs.(3) yield for the magnitude of $\delta_i(t + 1)$:

$$|\delta_i(t + 1)| = \min\{|x(t + 1) - x(t) + \delta_i(t)|, \Delta_i\} \quad (4)$$

The sign of $\delta_i(t + 1)$ is provided for by Eqs.(3). Yet, for $\delta_i(t) = \delta_i(t + 1) = \Delta_i$ [resp: $\delta_i(t) = \delta_i(t + 1) = -\Delta_i$], then [by way of Eq.(4)] $x(t + 1) - x(t) \geq 0$ [resp: $x(t + 1) - x(t) \leq 0$]. Hence, within the slip regime $\text{sgn}[x(t + 1) - x(t) + \delta_i(t)] = \text{sgn}[\delta_i(t)]$. Hence, the direction (extension or compression) of the $i$-th spring deformation (within the slip or stick regimes) may be incorporated into Eq.(4) as follows: State Equation (for each $i$):

$$\delta_i(t + 1) = \text{sgn}[x(t + 1) - x(t) + \delta_i(t)] \cdot \min\{|x(t + 1) - x(t) + \delta_i(t)|, \Delta_i\} \quad (5)$$

Distinguishing the slip regime as positive or negative, each operator may be thought of as having three distinct operational regimes:

(i) Positive slip: $\delta_i(t) = \Delta_i$

(ii) Stick: $|\delta_i(t)| < \Delta_i$
(iii) Negative slip: $\delta_i(t) = -\Delta_i$

It should be noted that transitions from any regime to any other are possible, depending upon the exerted displacement.

The Maxwell Slip model provides the total presliding friction force, $F_M(t)$, as the sum of the operators’ forces:

**Output Equation (sum of spring forces):**

$$F_M(t) = \sum_{i=1}^{M} k_i \cdot \delta_i(t)$$  \hspace{1cm} (6)

with the deformations $\delta_i(t)$ ($\forall i = 1, \ldots, M$) evolving according to Eq.(5).

In the sequel the input-output mapping provided by a Maxwell Slip model characterized by $M$ operators [Eqs.(5),(6)] will be designated as $F_M(t) = \Psi_{\delta_1}[M, k, d, x(t)]$, with $\delta_1$, $k$ and $d$ being the $M$-dimensional initial deformation vector, stiffness vector, and threshold vector, respectively:

$$\delta_1 \triangleq [\delta_1(1) \ldots \delta_M(1)]^T, \quad k \triangleq [k_1 \ldots k_M]^T, \quad d \triangleq [\Delta_1 \ldots \Delta_M]^T$$

The Maxwell Slip model structure is subject to the following assumptions:

**Assumption A1.** The operator stiffness values are nonzero: $k_i \neq 0$ ($\forall i$).

**Assumption A2.** The operator thresholds are strictly positive and distinct, say: $0 < \Delta_1 < \Delta_2 < \ldots < \Delta_M$.

The Maxwell Slip model has the capability of describing the hysteretic relationship between the displacement (excitation) and friction force (response) in the presliding friction regime [15, 16]. Typical examples of the hysteretic behavior obtained from a single operator and a multi operator Maxwell Slip model are illustrated in Figures 2 and 3, respectively. The former presents the $\delta_1(t)$ evolution (for $\Delta_1 = 0.5$) when the exerted displacement $x(t)$ varies linearly between two extreme values. The latter corresponds to a Maxwell Slip model with $M = 10$ operators with exerted displacement: $x(t) = \sin(0.2\pi t) \cdot \sin(5\pi t)$.

A number of useful definitions follow:

**Definition 2.1 (Slip border crossing)** Assume that there are two successive time instants, $t_0$ and $t_0 + 1$, such that $|\delta_i(t_0)| < \Delta_i$ and $\delta_i(t_0 + 1) = \Delta_i$ [resp: $\delta_i(t_0 + 1) = -\Delta_i$]. It is then said that $i$-th state variable $\delta_i(t)$ crosses its positive [resp: negative] slip border at time $t_0 + 1$.

\footnote{Bold face lower/upper case symbols designate vector/matrix quantities, respectively.}
Definition 2.2 (Stick border crossing) Assume that there are two successive time instants, \( t_0 \) and \( t_0 + 1 \), such that \( \delta_i(t_0) = \Delta_i \) [resp: \( \delta_i(t_0) = -\Delta_i \)] and \( |\delta_i(t_0 + 1)| < \Delta_i \). It is then said that the \( i \)-th state variable \( \delta_i(t) \) crosses its positive [resp: negative] stick border at time \( t_0 + 1 \).

Definition 2.3 (Slip direction change) Assume that there are two successive time instants, \( t_0 \) and \( t_0 + 1 \), such that \( \delta_i(t_0) = \Delta_i \) [resp: \( \delta_i(t_0) = -\Delta_i \)] and \( \delta_i(t_0 + 1) = -\Delta_i \) [resp: \( \delta_i(t_0 + 1) = \Delta_i \)]. It is then said that the \( i \)-th state variable \( \delta_i(t) \) changes slip direction at time \( t_0 + 1 \).

As the above definitions imply, there are three general types of operational regime transitions: slip border crossing, stick border crossing, and slip direction change.

Definition 2.4 (Reachability) The \( i \)-th operator is reachable iff its state variable \( \delta_i(t) \) may reach any point \( p \in S_i \), starting from any arbitrary point \( q \in S_i \) (\( S_i \) designating the interval \( [-\Delta_i, \Delta_i] \)).

2.2 Basic Properties

Property 2.1 Within the \( i \)-th operator’s stick regime, \( \delta_i(t) \) is a monotonically increasing function of \( x(t) \).

Proof: The proof is obvious from Eq.(5).

Property 2.1 is schematically depicted into Figures 4(a) and (b) for \( 5 \leq t < t' \).

Property 2.2 (Reachability) The \( i \)-th operator is reachable.

Proof: Considering Eq.(5), it is straightforward to construct at least one exerted displacement sequence that forces \( \delta_i(t) \) to reach any point \( p \in S_i \) starting from any arbitrary initial state.

Property 2.3 (Conditions for slip to slip/stick transitions.) Let \( t_0 \) and \( t_0 + 1 \) be two successive time instants with \( \delta_i(t_0) = \Delta_i \) [resp: \( \delta_i(t_0) = -\Delta_i \)], then:
(i) No change in slip direction:
\[ \delta_i(t_0 + 1) = \Delta_i \quad [\text{resp: } \delta_i(t_0 + 1) = -\Delta_i] \iff x(t_0 + 1) - x(t_0) \geq 0 \quad [\text{resp: } x(t_0 + 1) - x(t_0) \leq 0] \]

(ii) Slip direction change:
\[ \delta_i(t_0 + 1) = -\Delta_i \quad [\text{resp: } \delta_i(t_0 + 1) = \Delta_i] \iff x(t_0 + 1) - x(t_0) \leq -2\Delta_i \quad [\text{resp: } x(t_0 + 1) - x(t_0) \geq 2\Delta_i] \]

(iii) Stick border crossing:
\[ |\delta_i(t_0 + 1)| < \Delta_i \iff -2\Delta_i < x(t_0 + 1) - x(t_0) < 0 \quad [\text{resp: } 0 < x(t_0 + 1) - x(t_0) < 2\Delta_i]. \]

Proof:
(i) Since \( \delta_i(t_0) = \delta_i(t_0 + 1) = \Delta_i \) [resp: \( \delta_i(t_0) = \delta_i(t_0 + 1) = -\Delta_i \)], then [Eq.(5)] \( x(t_0 + 1) - x(t_0) + \Delta_i \geq \Delta_i \) [resp: \( x(t_0 + 1) - x(t_0) - \Delta_i \leq -\Delta_i \)]. Thus, \( x(t_0 + 1) - x(t_0) \geq 0 \) [resp: \( x(t_0 + 1) - x(t_0) \leq 0 \)]. The equivalence is straightforward.

(ii) Let \( \delta_i(t_0) = \Delta_i \). Then [Eq.(5) for \( t = t_0 \)]:
\[ \delta_i(t_0 + 1) = -\Delta_i \iff \text{sgn}[x(t_0 + 1) - x(t_0) + \Delta_i] < 0 \quad \text{and} \quad |x(t_0 + 1) - x(t_0) + \Delta_i| \geq \Delta_i \quad (7) \]

thus, \( \delta_i(t_0 + 1) = -\Delta_i \iff x(t_0 + 1) - x(t_0) \leq -2 \cdot \Delta_i \). The proof for \( \delta_i(t_0) = -\Delta_i \) is similar.

(iii) Let \( \delta_i(t_0) = \Delta_i \), then [Eq.(5) for \( t = t_0 \)]:
\[ |\delta_i(t_0 + 1)| < \Delta_i \iff |x(t_0 + 1) - x(t_0) + \Delta_i| < \Delta_i \iff -2 \cdot \Delta_i < x(t_0 + 1) - x(t_0) < 0 \quad (8) \]
The proof for \( \delta_i(t_0) = -\Delta_i \) is similar. \( \square \)

In case \( \delta_i(t) \) is within the operator’s positive [resp: negative] slip regime then, according to Property 2.3(i), it remains within this regime iff \( x(t) \) get increased [resp: decreased] (see Figures 4(a) and (b) for \( t' \leq t \leq t_0 \)). It leaves the regime [changes slip direction or crosses stick border - Properties 2.3(ii) – (iii)] only after \( x(t) \) starts to decrease [resp: increase]. Therefore, slip direction changes or stick border crossings [Properties 2.3 (ii) and (iii)] occur right after \( x(t) \) attains a local extremum (in other words at velocity reversals). This is depicted in Figure 4, in which \( x(t) \) reaches a local maximum at \( t_0 \) [Figure 4(a)], \( \delta_1(t) \) changes slip direction at \( t_0 + 1 \) [Figure 4(b)] and \( \delta_2(t) \) crosses its positive stick border at \( t_0 + 1 \) [Figure 4(c)]. Notice that \( \Delta_1 < \Delta_2 \).

**Property 2.4** *(Operator slip to stick and/or slip direction changes occur simultaneously)* Let a Maxwell Slip model with \( M \) operators and a time instant \( t_0 \) such that \( \delta_i(t_0) = \Delta_i \) [resp: \( \delta_i(t_0) = -\Delta_i \)], \( \forall i = \)
1, . . . , M. Then δi(t0 + 1) ≠ ∆i [resp: δi(t0 + 1) ≠ −∆i], ∀i = 1, . . . , M iff at least one of them, say δj(t0 + 1) ≠ ∆j [resp: δj(t0 + 1) ≠ −∆j]. □

Proof: Since δj(t0) = ∆j and δj(t0 + 1) ≠ ∆j, then δj(t) crosses its positive stick border or changes slip direction at t = t0 + 1. Thus, x(t0 + 1) − x(t0) < 0 [Properties 2.3 (ii) and (iii)]. Since all operators are subject to the same x(t), and δi(t0) = ∆i, ∀i = 1, . . . , M, then [Eq.(5) for t = t0]:

δi(t0 + 1) = sgn[x(t0 + 1) − x(t0) + ∆j] · min{|x(t0 + 1) − x(t0) + ∆j, ∆i}| < ∆i, ∀i = 1, . . . , M

Thus δi(t0 + 1) ≠ ∆i, ∀i = 1, . . . , M. The proof for δi(t0) = −∆i, ∀i = 1, . . . , M, is similar. □

Property 2.4 implies that when the operators of a Maxwell Slip model are within their slip regime, the transition of one operator to a different regime (stick or opposite slip) forces the simultaneous transition of all operators. Each operator may thus change slip direction or stick, depending upon the conditions in Properties 2.3 (ii) and (iii), respectively. This behavior is also schematically depicted at Figures 4(b) and (c), respectively. Notice also that in case that both δj(t0) = ∆j and δi(t0) = ∆i [resp: δj(t0) = −∆j and δi(t0) = −∆i], with ∆j < ∆i, then if δj(t) crosses its positive [resp: negative] stick border at t0 + 1, then δi(t) crosses its positive [resp: negative] stick border at t0 + 1 as well [Property 2.3 (iii) - see Figure 5(b) at time tmax + 1 (∆1 < ∆2)].

Property 2.5 (Conditions for stick to slip transitions) Let |δi(t)| < ∆i, ∀t ∈ (t0, ti]. Then:

(i) δi(ti + 1) = ∆i ⇐⇒ x(ti + 1) − x(t0) ≥ ∆i − δi(t0) (positive slip border crossing)

(ii) δi(ti + 1) = −∆i ⇐⇒ x(ti + 1) − x(t0) ≤ −∆i − δi(t0) (negative slip border crossing) □

Proof: Since |δi(t)| < ∆i, ∀t ∈ (t0, ti], then the evolution of δi(t) is [Eq.(5)]:

δi(t0 + 1) = x(t0 + 1) − x(t0) + δi(t0), for t = t0

δi(t0 + 2) = x(t0 + 2) − x(t0 + 1) + δi(t0 + 1) = x(t0 + 2) − x(t0) + δi(t0), for t = t0 + 1

...  

δi(t + 1) = x(t + 1) − x(t0) + δi(t0), ∀t ∈ (t0, ti − 1]  

...  

δi(ti) = x(ti) − x(t0) + δi(t0), (t = ti − 1)  

(9) ...

(10)
For \( \delta_i(t_i + 1) = \Delta_i \), Eq.(5) gives: \( x(t_i + 1) - x(t_i) + \delta_i(t_i) \geq \Delta_i \). Substituting \( \delta_i(t_i) \) from Eq.(10) into this yields:

\[
x(t_i + 1) - x(t_0) \geq \Delta_i - \delta_i(t_0)
\]

(positive slip border crossing)

Similarly, for \( \delta_i(t_i + 1) = -\Delta_i \), Eq.(5) gives: \( |x(t_i + 1) - x(t_i) + \delta_i(t_i)| \geq \Delta_i \iff -x(t_i + 1) + x(t_i) - \delta_i(t_i) \geq \Delta_i \). Substituting \( \delta_i(t_i) \) from Eq.(10) into this yields:

\[
x(t_i + 1) - x(t_0) \leq -\Delta_i - \delta_i(t_0)
\]

(negative slip border crossing)

\[\square\]

Assuming that \( t_0 \) is the initial time instant \( (t_0 = 1) \), then Property 2.5 demonstrates how the initial state influences the time appearance of the first slip border crossing (see Figures 5(a) and (c) at time \( t_2 \)). Similarly, if \( |\delta_i(t_0)| = \Delta_i \) and \( |\delta_i(t)| < \Delta_i, \forall t \in (t_0, t_i] \), then for \( t = t_0 \) a local extremum (maximum or minimum) of \( x(t) \) has been obtained [stick border crossing at \( t = t_0 + 1 \) - see Property 2.3\( (iii) \]). Thus, it is demonstrated [Eq.(9)] how the evolution of \( \delta_i(t) \) within the operator’s stick regime, as well as its next slip border crossing, gets influenced by \( \Delta_i \), the previous local extremum (this that corresponds to the previous stick border crossing) and the current value of \( x(t) \).

**Property 2.6** Let \( \delta_i(t_0) = \Delta_i \) and \( \delta_j(t_0) = \Delta_j \) (resp: \( \delta_i(t_0) = -\Delta_i \) and \( \delta_j(t_0) = -\Delta_j \), with \( \Delta_i < \Delta_j \), and \( x(t) \) decreasing (resp: increasing) for \( t > t_0 \), with \( -2\Delta_i < x(t_0 + 1) - x(t_0) \) (resp: \( x(t_0 + 1) - x(t_0) < 2\Delta_i \)). If \( \delta_j(t) \) crosses its negative (resp: positive) slip border at time \( t_j > t_0 \), then \( \delta_i(t) \) has already crossed its negative (resp: positive) slip border.

**Proof:** Since \( \delta_i(t_0) = \Delta_i, \delta_j(t_0) = \Delta_j \) and \( x(t) \) decreases for \( t > t_0 \), with \( -2\Delta_j < -2\Delta_i < x(t_0 + 1) - x(t_0) \) \( [\Delta_i < \Delta_j] \), then both state variables cross their positive stick border at time \( t_0 + 1 \) (Properties 2.3\( (iii) \) and 2.4). Notice that they may remain stuck for some time.

Then, let \( \delta_j(t) \) cross its negative slip border at a later time \( t_j \). Thus \( x(t_j) - x(t_0) \leq -2\Delta_j \) (Property 2.5). Since \( \Delta_i < \Delta_j \), then \( x(t_j) - x(t_0) \leq -2\Delta_j < -2\Delta_i \). Therefore, due to Property 2.5, \( \delta_i(t_j) = -\Delta_i \).

The proof for \( \delta_i(t_0) = -\Delta_i \) and \( \delta_j(t_0) = -\Delta_j \) is similar.

\[\square\]

A schematic representation of Property 2.6 is depicted at Figure 5(b). Notice that \( \delta_2(t) \) crosses its negative slip border at time \( t' \) and \( \Delta_1 < \Delta_2 \).
Property 2.7 (Transient response) Assume an operator with state evolution starting with the $\delta_1(1)$ initial state. The transient response due to this initial state disappears right after the first slip border crossing.

Proof: This is evident as $|\delta_1(t)| = \Delta_i$ at the first slip border crossing.

Property 2.7 is schematically depicted into Figure 5(c), in which the evolutions of $\delta_2(t)$, starting from different initial states $\bar{\delta}_2(1)$ and $\tilde{\delta}_2(1)$, coincide after time $t_2$.

3 Maxwell Slip Model Identifiability

Identifiability is a fundamental issue for successful model identification and incorporates the concepts of a priori identifiability and a posteriori identifiability. The former examines the question of whether or not a postulated model structure may be uniquely identified from a designed experiment, under the ideal conditions of noise free observations and model structure correctness [18] (uniqueness property). The latter deals with whether a real data set is “informative enough” to allow for the discrimination of a particular model from the pool of models belonging to the model structure [19].

Although a priori identifiability is necessary, it is not sufficient for guaranteeing a posteriori identifiability [18]. The form of excitation employed has an apparently fundamental role within this context.

3.1 A Priori Identifiability of the Maxwell Slip Model Structure

Maxwell Slip model identification in general deals with the estimation of the initial state vector $\delta_1$ (which is unavailable), stiffness vector $k$ and threshold vector $d$ from available excitation-response (displacement - presliding friction force) data.

Definition 3.1 (A priori identifiability)[20] The Maxwell Slip model structure is a priori identifiable with respect to $k$ and $d$, from excitation - response data iff equality of two models (characterized by $k$, $d$, $\delta_1$ and $\bar{k}$, $\bar{d}$, $\bar{\delta}_1$, respectively) is equivalent to equality of the $k$ and $d$ vectors, that is:

$$\Psi_{\delta_1}[M, k, d, x(t)] = \Psi_{\bar{\delta}_1}[M, \bar{k}, \bar{d}, x(t)] \iff k = \bar{k}, \ d = \bar{d}$$

(11)

for at least one excitation sequence $x(t)$ and for each common initial state vector $\delta_1 = \bar{\delta}_1$. 

□
Proposition 3.1 (A priori identifiability of $k$ and $d$) The Maxwell Slip model structure, subject to assumptions A1-A2, is a priori identifiable with respect to the $k$ and $d$ vectors. □

Proof: See Appendix A. □

Proposition 3.2 (Lack of a priori identifiability for the initial state vector $\delta_1$) The Maxwell Slip model structure, characterized by two or more operators and subject to assumptions A1-A2, is not a priori identifiable with respect to the initial state vector $\delta_1$. □

Proof: See Appendix B. □

3.2 A Posteriori Identifiability of the Maxwell Slip Model Structure

In order for a posteriori identifiability, with respect to $k$ and $d$, to be obtained, the excitation should provide for clear discrimination among models belonging to the a priori identifiable (with respect to $k$ and $d$) Maxwell Slip model structure. The issue is examined in the following three subsections: The first is devoted to initial state effects, the second to a posteriori identifiability, and the third to excitation design.

3.2.1 Initial state effects

Property 2.5 states that the initial states, $\delta_i(1) \ [\forall i = 1, \ldots, M]$, along with the exerted excitation $x(t)$ and the thresholds $\Delta_i$, determine the time of the first slip border crossing for each operator [each $\delta_i(t)$]. Evidently, and in view of Proposition 3.2, incorrect estimation of one or more initial states may lead to incorrect determination of the time of the first slip border crossing for that or those operators [see Figure 5(c)].

Consequently, in order to avoid problems with the identification due to misspecified initial states, their effects (transients) should be eliminated. A way to do is by taking advantage of Property 2.7, which indicates that the initial state effects disappear right after the first slip border crossing. This leads to the following condition for the exerted excitation:

Condition on the exerted excitation (condition $C0$): The excitation $x(t)$ needs to be designed such that it gets increased (or decreased), in an arbitrary way, up to a critical maximum (or minimum)
value $x_{cr} = x(t_{cr})$. This value should be such that all operators are forced to slip (this is feasible due to Properties 2.1 and 2.2). This eliminates transient effects, and the data corresponding to $t \geq t_{cr}$, combined with $\delta_i(t_{cr}) = \text{sgn}[x(t_{cr})] \cdot \Delta_i$, $\forall i = 1, \ldots, M$, may be used for identification.

### 3.2.2 A posteriori identifiability

As is also indicated in the proof of Proposition 3.1 (Appendix A), the identifiability of the Maxwell Slip model is closely related to the type of operational regime transitions enforced by the excitation $x(t)$. In the multi-operator Maxwell Slip model case there can be several potential combinations of such transitions that suffice for a posteriori identifiability. Nevertheless, the existence of operators that undergo only slip direction transitions is undesirable, as such behavior hinders information relating to their stick regime to be extracted. In general, the smaller the threshold of a state variable, the greater the possibility of exhibiting such behavior [see Property 2.3(ii)]. For this reason the concept of Lower Threshold Limit (LTL) is presently introduced.

**Definition 3.2 (Lower Threshold Limit – LTL)** Assume a given excitation $x(t)$ and a Maxwell Slip model with thresholds $\Delta_i$ ($i = 1, 2, \ldots, M$). The LTL is defined as that quantity, entirely depending upon the given excitation, for which the operators with thresholds smaller or equal to it undergo no stick border crossing for the given $x(t)$. That is:

$$\Delta_i \leq \text{LTL} \implies \text{the } i\text{-th operator undergoes no stick border crossing} \quad (12)$$

**Proposition 3.3 (LTL determination)** Given an exerted displacement sequence $x(t)$, the Lower Threshold Limit (LTL) is determined as:

$$\text{LTL} = \frac{1}{2} \cdot \min_{t_e} |x(t_e + 1) - x(t_e)| \quad (13)$$

with $t_e$ designating a time instant at which the exerted displacement sequence exhibits a local extremum.

**Proof:** The proof is a direct consequence of Properties 2.3 (ii) and (iii). First observe that an operator leaves its slip regime right after a local extremum of the excitation $x(t)$. Then [Property 2.3(iii)] in
order for the operator with the smallest threshold (by convention operator 1 with threshold $\Delta_1$) to have the possibility to experience (at least one) stick border crossing, the minimum [over all time instants associated with a local extremum of the excitation $x(t)$] absolute difference $|x(t_{e+1}) - x(t_{e})|$ must be smaller than twice the threshold.

The a posteriori identifiability is examined for the class $\mathcal{E}$ of excitation signals that attain a maximum [resp: minimum] value $x_{cr} = x(t_{cr})$ (and thus satisfy condition C0) and consequently decrease [resp: increase] monotonically. This class of signals is important for their simplicity and ease of implementation.

**Proposition 3.4 (A posteriori identifiability of the Maxwell Slip model structure)** Let a Maxwell Slip model with $M$ operators, subject to Assumptions A1 – A2, operating under an excitation that belongs to the aforementioned class $\mathcal{E}$. Assume, without loss of generality that $x(t_{cr})$ is a positive excitation maximum. A set of necessary and sufficient conditions on the excitation for achieving a posteriori identifiability with respect to the $k$ and $d$ vectors then is:

(C1) $-2\Delta_1 < x(t_{cr} + 1) - x(t_{cr}) < 0$

(C2) $\forall j = 2, \ldots, M$ there is at least one $t_j > t_{cr} + 1$ such that:

$$-2\Delta_j < x(t_{j+1}) - x(t_{cr}) < x(t_{j}) - x(t_{cr}) \leq -2\Delta_{j-1}$$

with $0 < \Delta_1 < \Delta_2 < \ldots \Delta_M$ [conditions C1 and C2 should be properly adjusted for $x(t_{cr})$ being a minimum].

**Proof:** See Appendix C.

Since $\Delta_1$ is the smallest threshold, condition C1 implies that $-2\Delta_M < \ldots < -2\Delta_2 < -2\Delta_1 < x(t_{cr} + 1) - x(t_{cr}) < 0$, thus indicating that all operators cross their positive stick border at time $t_{cr} + 1$ (no slip direction changes are exhibited; see Properties 2.3 and 2.4). As demonstrated in the proof of Proposition 3.4, failure of an operator to do so would jeopardize a posteriori identifiability.

Condition C2 implies that as $x(t)$ decreases (for $t > t_{cr}$), the operators cross their corresponding negative slip borders sequentially (following their simultaneous positive stick border crossing at time $t_{cr} + 1$), in such a way that $\delta_j(t) \ [\forall j = 2, \ldots, M]$ remains sticking for at least two time instants after $\delta_{j-1}(t)$ crosses its negative slip border. Notice that if $\delta_{j-1}(t)$ remains sticking then $\delta_j(t)$ is impossible to be within its negative slip regime (assumption A2 and Property 2.6).
Condition $C2$ may be violated in two possible ways by at least one operator $j \in [2, M]$:

(a) $\delta_{j-1}(t)$ and $\delta_j(t)$ cross their negative slip border at the same time instant $t_j > t_{cr} + 1$. Then (Property 2.5):

$$x(t_j) - x(t_{cr}) \leq -2\Delta_j < -2\Delta_{j-1} < x(t_j - 1) - x(t_{cr})$$  \hspace{1cm} (14)

(b) $\delta_{j-1}(t)$ and $\delta_j(t)$ cross their negative slip border at successive time instants $t_j$ and $t_j + 1$, with $t_j + 1 > t_j > t_{cr} + 1$. Then (Property 2.5):

$$x(t_j + 1) - x(t_{cr}) \leq -2\Delta_j < x(t_j) - x(t_{cr}) \leq -2\Delta_{j-1} < x(t_j - 1) - x(t_{cr})$$  \hspace{1cm} (15)

Expressions (14), (15) imply that if there are successive thresholds $\Delta_{j-1}$ and $\Delta_j$ that satisfy any one of the following relationships:

$$\Delta_{j-1}, \Delta_j \in \left[ \frac{x(t_{cr}) - x(t) - 1}{2}, \frac{x(t_{cr}) - x(t)}{2} \right], \text{ with } \Delta_{j-1} \neq \Delta_j, \text{ for } t > t_{cr} + 1$$  \hspace{1cm} (16)

$$\Delta_{j-1} \in \left[ \frac{x(t_{cr}) - x(t) - 1}{2}, \frac{x(t_{cr}) - x(t)}{2} \right], \Delta_j \in \left[ \frac{x(t_{cr}) - x(t) + 1}{2}, \frac{x(t_{cr}) - x(t + 1)}{2} \right] \text{ for } t > t_{cr} + 1$$  \hspace{1cm} (17)

the a posteriori identifiability condition $C2$ is violated (loss of a posteriori identifiability).

Viewed in a slightly different way, condition $C2$ implicitly states that successive thresholds which lie within the ranges prescribed by Eq.(16) or Eq.(17) cannot be discriminated by the given excitation $x(t)$. Observe that the aforementioned ranges depend exclusively upon the excitation $x(t)$ and get narrower as $x(t)$ gets smoother.

### 3.2.3 Excitation design

The guidelines for designing a proper excitation signal $x(t)$ are provided by conditions $C0$, $C1$ and $C2$. Based upon them, a procedure for designing a very simple excitation that is sufficiently rich for identification may be described as follows.

Let, at first, $x(t)$ increase (linearly or otherwise) up to a value $x_{cr} = x(t_{cr})$ that is sufficient for getting all operators slipping (condition $C0$). The data to be used in the identification correspond to the time instant $t_{cr}$ forward. Next, let $x(t)$ get slowly and monotonically decreased (linearly or otherwise) up to a
point \( x'_{cr} = x(t'_{cr}) \) in such a way as to meet conditions \( C1 \) and \( C2 \). Let \( x(t) \) subsequently vary (linearly or otherwise) up and down within the \([x_{cr}, x'_{cr}]\) interval.

This design involves the selection of \( x_{cr} \), \( x'_{cr} \) and the form of the excitation variation within these bounds (hereby referred to as the “excitation smoothness factor”):

Selecting \( x_{cr} \) and \( x'_{cr} \): A potential choice for \( x_{cr} \) and \( x'_{cr} \) may be the displacements exerted right before positive-direction and negative-direction, respectively, gross sliding begins.

Smoothness factor of \( x(t) \): The required excitation smoothness is closely related to conditions \( C1 \) and \( C2 \). The former condition implies that the decrement \( x(t_{cr} + 1) - x(t_{cr}) \) dictates the smallest threshold \( \Delta_1 \) that may be identified. The latter condition dictates that:

(i) How close two successive thresholds can be in order to be identifiable [Eqs.(16) - (17)].

(ii) The second largest threshold that may be identified: \( \Delta_{M-1} \leq [x(t_{cr}) - x(t'_{cr} - 1)]/2 \) (see condition \( C2 \) for \( j = M \) and \( t_{M} = t'_{cr} - 1 \)).

Notice that the maximum threshold may be arbitrarily large: \( \Delta_{M} > [x(t_{cr}) - x(t'_{cr})]/2 \) (see condition \( C2 \) for \( j = M \) and \( t_{M} + 1 = t'_{cr} \)).

4 Maxwell Slip Model Structure Based Presliding Friction Identification

In this section two new methods for the Maxwell Slip model structure based identification of presliding friction dynamics are postulated: The Dynamic Linear Regression (DLR) and the NonLinear Regression (NLR) methods. They may be both thought of as different extensions of the conventional Linear Regression (LR) method that uses (arbitrary) threshold (maximum spring deformations \( \Delta_i \')s \) preassignment [15, 16]): The DLR by introducing extra dynamics in the form of a vector finite impulse response filter, and the NLR by relaxing threshold preassignment through a special nonlinear regression procedure.

All methods, including the conventional LR which is presented first for purposes of completeness, on one hand, but also for forming the basis for the introduction of the DLR and NLR methods, on the other, are based upon minimization of a quadratic cost function of the form:

\[
\mathcal{J} \triangleq \sum_{t=1}^{N} e^2(t)
\]
with \( N \) designating the number of excitation–response signal samples used in identification, and \( e(t) \) the error defined as the difference between the measured force \( F(t) \) and the model provided force \( F_M(t) \):

\[
e(t) = F(t) - F_M(t)
\]  

(19)

As is conventionally done, this error is assumed to be a stationary zero mean and uncorrelated (white) sequence with variance \( \sigma^2_e \) [19] (note that relaxation of the error assumptions is generally possible, but is beyond the scope of the present study).

### 4.1 The Linear Regression (LR) Method

The conventional Linear Regression (LR) is the simplest possible method accomplishing partial identification of the Maxwell Slip model [15, 16]. The nonlinear part of the model [Eq.(5)] is assumed known, as the thresholds (\( \Delta_i \)'s) are preassigned, while only the remaining linear part [Eq.(6)] is estimated. As the presliding friction regime is considered, the thresholds are (arbitrarily) uniformly preassigned as:

\[
\Delta_i = \frac{i}{M} \times \max_t \{|x(t)|\} \quad (i = 1, \ldots, M)
\]  

(20)

The presliding friction force is then represented via an LR\( (M) \) model (\( M \) designating the number of elasto-slide operators) of the form [compare to Eq.(6)]:

\[
\text{LR}(M, k): \quad F(t) = \sum_{i=1}^{M} k_i \cdot \delta_i(t) + e(t)
\]  

(21)

subject to Eq.(5). As the model structure is linear in the parameters, minimization of the cost function (18) leads to a Linear Regression type estimator for \( k \).

### 4.2 The Dynamic Linear Regression (DLR) Method

This DLR method is based upon a suitable extension of the Maxwell Slip model structure. Once again, partial identification is performed, as the nonlinear part of the model [Eq.(5)] is assumed known (the thresholds being preassigned). Nonetheless, the linear part of the original model is extended by having the presliding friction force depending not only upon present, but also past values of the spring deformations [\( \delta_i(t) \)'s]. This is accomplished by having each spring deformation driven through a Finite Impulse Response (FIR) filter of order \( n \) in producing the presliding friction force. Equivalently, the
spring deformation vector $\delta(t)$, defined as:

$$
\delta(t) \triangleq [\delta_1(t) \ldots \delta_M(t)]^T
$$

is driven through an $M$-dimensional FIR filter with vector coefficients $\theta_i$ ($i = 0, \ldots, n$).

The DLR($M,n,\theta$) model is thus of the form:

$$
\text{DLR}(M,n,\theta): \quad F(t) = \sum_{i=0}^{n} \theta_i^T \cdot \delta(t - i) + e(t)
$$

subject to Eq.(5). The vector $\theta$ used in the model representation designates the composite FIR filter parameter vector (which is to be estimated) and $e(t)$ the model error.

Observe that the DLR model structure conserves linearity in the parameters, hence minimization of the cost function of Eq.(18) still leads to a Linear Regression type estimator for $\theta$. The main advantage of this model is the additional dynamics and flexibility introduced via the vector FIR filter. This may (to a certain extent) account for discrepancies between the Maxwell Slip model and the actual friction dynamics that may be due to either false threshold preassignment and/or the presence of otherwise unaccounted dynamics.

### 4.3 The NonLinear Regression (NLR) Method

The NonLinear Regression (NLR) method aims at complete estimation of the Maxwell Slip model, that is estimation of both the thresholds and stiffnesses. The NLR($M$) model thus is of the form:

$$
\text{NLR}(M,k,d): \quad F(t) = \sum_{i=1}^{M} k_i \cdot \delta_i(t) + e(t)
$$

subject to Eq.(5), with $e(t)$ designating the model error.

The relaxation of the threshold preassignment is naturally expected to lead to increased accuracy. Yet, the price to be paid for this benefit is that linearity in the model parameters is lost. Therefore, minimization of the cost function $J$ [Eq.(18)] now leads to a NonLinear Regression type estimator. By observing that the model is nonlinear only with respect to the threshold vector $d$, while remaining linear with respect to the stiffness vector $k$, the estimator may be realized via a succession of nonlinear and linear regression operations, that is$^2$:

$$
\left[ k^T d^T \right]^T = \arg \min_{k,d} J(k,d) = \arg \min_{d} \{ \min_{k} J(k/d) \}
$$

$^2$The hat designates estimator/estimate.
The nonlinear regression operation is based upon a postulated two-phase, hybrid, optimization scheme. The first (pre-optimization) phase utilizes Genetic Algorithm (GA) based optimization [21] in order to explore large areas of the parameter space and locate regions where global or local minima may exist. The second (fine-optimization) phase utilizes the Nelder-Mead Downhill Simplex algorithm [22] for locating the exact global or local minima within the previously obtained regions.

This two-phase scheme has been shown (see section V) to be effective in locating the true global minimum of the cost function and circumventing problems associated with local minima, which are quite severe in this case [see, for instance Figure 6 which presents the contour plot of the cost function for an NLR(1) model]. Furthermore, the Nelder-Mead algorithm used utilizes only cost function evaluations and no derivatives, which are not defined everywhere as the cost function is nonsmooth in areas of the parameter space.

**Remark:** In case that identification of the initial state vector, \( \delta_1 \), along with the stiffness and threshold vectors \( k \) and \( d \), respectively, is attempted, the nonlinear regression problem is somewhat reformulated and the number of unknowns increases from \( 2M \) to \( 3M \). Since the model is nonlinear with respect to initial state vector \( \delta_1 \) as well, its estimation is performed by the nonlinear regression operation. Yet, the initial states are bounded by the corresponding thresholds \( (|\delta_i(1)| \leq \Delta_i, \forall i = 1, \ldots, M \) – see subsection 2.1). A way of incorporating this into the the NLR method is to set \( \delta_i(1) = \Delta_i \cdot \sin(\phi_i), \) with \( \phi_i \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), \( \forall i = 1, \ldots, M \), and perform the nonlinear regression with respect to \( \Delta_i \) and \( \phi_i \), \( \forall i = 1, \ldots, M \). The interval choice is guided by the fact that \( \forall \phi_i, \phi^*_i \in [-\frac{\pi}{2}, \frac{\pi}{2}], \phi_i \neq \phi^*_i \iff \sin(\phi_i) \neq \sin(\phi^*_i) \).

### 4.4 Model Order Selection

Since the main objective of presliding friction identification is simulation and control, model selection is tailored to these needs and is primarily judged in terms of the identified model’s simulation ability. This is measured via a normalized quadratic function of the model error, referred to as the *Normalized Output Error (NOE)*:

\[
NOE = \frac{\sum_{t=1}^{N}(F(t) - F_M(t))^2}{\sum_{t=1}^{N}(F(t) - \bar{\mu}_F)^2} \times 100%
\]

where \( F(t) \) and \( F_M(t) \) designate the actual friction and its model-based simulated counterpart, respectively, \( \bar{\mu}_F \) the sample mean of the actual friction force, and \( N \) the signal length. An advantage of this criterion is that it focuses on the excitation–response dynamics and is applicable to any model structure.
In addition, the condition number of the information matrix [19] is used as a means of avoiding model overfitting.

LR(M) and NLR(M) model order selection (that is selecting the number M of superimposed operators) is based upon the successive estimation of models for increasing M and evaluation of the error criterion. On the other hand, DLR(M, n) model order selection is based upon the estimation of models corresponding to various values of n for any given M. The final model is selected following consideration of various values of M. Model assessment is based upon a model’s simulation performance examination within a subset of the data, referred to as the validation set, that have not been used in the estimation (cross validation principle).

5 Identification Results

Identification results with two different types, referred to as Data Sets A and B, of excitation–response (displacement–friction force) data sets are presented.

Data Set A includes 30 excitation – response realizations (Monte Carlo runs) obtained via a Maxwell Slip model. The aim of this investigation is to explore the capabilities and performance characteristics of the methods (mainly the LR and NLR, as the DLR employs an essentially different model structure) for accurate Maxwell Slip model parameter estimation.

Data Set B includes experimental displacement – friction data obtained from a physical system (two surfaces in contact) which operates within the presliding friction regime. This data set has been provided by the Katholieke Universiteit Leuven research team led by Professors H. Van Brussel and F. Al-Bender. The objective is the assessment of the capabilities of the three identification methods to in accurately describing the underlying dynamics.

Before proceeding with detailed descriptions and results, certain details concerning the Genetic Algorithm based optimization used within the context of the NLR method are provided. Each threshold is Gray coded [23] using 20 bits over the interval [LB, UB]. The lower bound, LB, is set equal to LTL, while the upper bound is arbitrarily chosen. The Genetic Algorithm incorporates: (i) a nonlinear ranking operator with selective pressure equal to 1.3, (ii) a stochastic universal sampling operator with generation gap equal to 0.97, (iii) a two-point crossover operator with crossover probability equal to
0.80, (iv) a mutation operator with mutation probability equal to 0.01 and (v) a fitness-based reinsertion operator. The population size and number of generations are selected according to model complexity (not exceeding 300). In case of initial state identification, each \( \phi_i \) is also Gray coded using 20 bits over the interval \([\frac{-\pi}{2}, \frac{\pi}{2}]\).

5.1 Data Set A

Data Set A includes realizations includes 30 different excitation–response realizations obtained via a Maxwell Slip model characterized by \( M = 3 \) operators. The model parameters (stiffnesses and thresholds) are indicated in Table 1. Each excitation realization (displacement) is a zero mean uncorrelated Gaussian distributed random signal consisting of \( N = 1,000 \) samples. The responses are either noise free or corrupted by zero mean uncorrelated noise at the 5\% standard deviation level.

5.1.1 No noise case

The ability of the LR and NLR methods to provide accurate estimates of the actual Maxwell Slip model parameters is investigated via Monte Carlo experiments (30 runs per case) using the correct model order (\( M = 3 \)). Three types of results are presented:

(i) Assuming zero initial states (Type I results for LR and NLR).

(ii) Estimating initial states (Type II result for NLR).

(iii) Eliminating initial state effects via condition \( C_0 \) (Type III results for LR and NLR).

The estimated model parameters results are contrasted to the true parameters in Table 1. The Normalized Output Errors (NOE’s) attained in each case are also presented. Examination of the results indicates that, as expected, threshold preassignment leads to biased estimates of the stiffness parameters for the LR(3) models (Type I and III results). On the contrary, the NLR(3)–based stiffness and threshold estimates are excellent, coinciding with their actual counterparts (results of all Types). Nevertheless, the initial state estimates (Type II result) are totally inaccurate, confirming the earlier stated fact on the lack of identifiability with respect to the initial state (Proposition 3.2).

The standard deviations of the NLR(3)–based stiffness and threshold parameter estimates (Type I and II results) are significantly higher than those of the Type III results (the NOE in the last case is almost
zero owing to the absence of added noise). The source of these discrepancies, for the type I results, is the erroneous preselection of the initial states, while for the type II results is the local minima in the cost function which are introduced due to the lack of identifiability of the initial states.

The overall best NOE criterion (with enormous differences from the rest) is thus achieved by the NLR(3) model for the Type III results. This underlines the effectiveness and significance of eliminating the initial state effects for achieving highly accurate stiffness and threshold estimates.

5.1.2 5% noise case

In this case both model order selection and parameter estimation are investigated with noise–corrupted response (force) signals. The initial state effects have been eliminated via condition $C_0$.

Model order selection results are, for all three methods, presented in Figure 7. DLR identification leads to generally decreasing NOE as the FIR order $n$ increases ($n = 1, \ldots, 10$). The FIR order $n$ beyond which the reduction in the NOE is practically insignificant (below 1%) is selected and presented in the DLR($M, n$) models in Figure 7.

The NOE, as a function of the number of operators $M$ and for the three types of models, is presented in Figure 7. It is worth observing that the NLR($M$) models uniformly achieve the best accuracy as their NOE values are significantly lower than those of their counterparts. Furthermore, a plateau in the NOE sequence is achieved for $M \geq 3$, hence the NLR(3) model is selected (notice that this model is characterized by the correct number of operators). The DLR($M, n$) models achieve the second best accuracy. The minimal NOE value in this case is achieved by the overdetermined DLR(5, 6) model. Finally, the LR($M$) models achieve the highest NOE values (with small differences from their DLR counterparts), with the overdetermined LR(5) model attaining the minimum.

Next, Monte Carlo estimation results (30 runs per case) for models characterized by $M = 3$ operators are presented in order to enable comparisons with the actual system. LR(3), DLR(3, 4) and NLR(3) results are pictorially presented in Figure 8, in which point estimates (middle horizontal line of each box) plus/minus two standard deviations (upper and lower horizontal lines of each box) are depicted. From these results the bias in the LR(3) stiffness estimates and the high accuracy of the NLR(3) model parameters are evident. The high quality of the NLR(3) model is reflected in the achieved NOE value which is significantly lower than those of the DLR(3, 4) and LR(3) models [Figure 8(a)].
5.2 Data Set B

Data Set B includes the 25,033 sample-long displacement – friction signals obtained from a laboratory device (contacting materials: brass sliding on thin plastic substrate backed by hard steel) which operates within the presliding friction regime. The exerted displacement is wideband random, and the signals are sampled at $f_s = 250 \text{ Hz}$. Both signals possess experimental offsets, due to sensor calibration, which are removed by extracting the corresponding sample means.

Initial state effects are eliminated as previously suggested, by locating the global extremum (minimum) of the excitation which appears at $t_{cr} = 18,252$. Thus, the remaining 6,781 samples are used in identification. They are divided into two disjoint sets: A 2,000 sample-long estimation set (for $t = 18,252 \ldots 20,251$) used for identification (including both parameter estimation and model order selection), and a 4,781 sample-long validation set (for $t = 20,252 \ldots 25,033$) used for independent evaluation and assessment of the estimated model performance.

Model order selection results by all three methods are presented in Figure 9, which depicts the Normalized Output Error (NOE) criterion as a function of the number of operators included in the model. Concerning the DLR method, Figure 9 presents the FIR order $n$ (for each $M$) beyond which the NOE decrease is practically insignificant. Figure 9(a) refers to the estimation set and Figure 9(b) to the validation set.

As Figure 9(a) indicates, the NOE criterion is, for all models, decreasing as the order increases. Nevertheless, the rate of NOE decrease for the LR models becomes small for $M \geq 6$, while for the NLR models the NOE reaches an approximate plateau for $M \geq 6$. Thus the LR(6) and NLR(6) models are selected. Regarding the DLR models, the NOE is decreasing for $M \leq 7$, remains almost unchanged for $M = 8$, and continues its decrease for $M \geq 9$. For purposes of model economy (principle of parsimony) the DLR(7, 8) model is in this case selected.

The behavior of the selected models within the validation set is analogous [Figure 9(b)]. This suggests that the estimated models may be considered as valid *approximate* representations of the presliding friction dynamics. However, for the NLR sequence of models, there appears to be an increase in the NOE criterion for $M \geq 9$ [Figure 9(b)]. Nevertheless, examination of the NLR(9) and NLR(10) models indicates that they incorporate operators with almost equal thresholds and opposite stiffnesses. Thus,
the fundamental assumption $A_2$ is violated. Furthermore, the individual responses of these operators are mutually cancelled (Property 2.7). Thus, these models are clearly overdetermined (probably in an attempt to account for friction dynamics not accounted for by the Maxwell Slip model structure).

The fact that all selected models appear capable of capturing, though at different degrees, the underlying presliding friction dynamics may be also confirmed by the results presented in Figure 10. This compares the actual presliding friction to the presliding friction obtained by driving each one of the estimated models [LR(6), DLR(7, 8) and NLR(6)] by the measured excitation (part of the validation set is shown). The corresponding model error signals are, for each model case, also presented.

A detailed comparative performance assessment, within the validation set, of the selected models is presented in Table 2. This shows the NOE criterion along with the corresponding absolute maximum error. The parametric complexity of each model (number of estimated parameters), along with the computational time required for its estimation, are also presented. As it may readily observed, the DLR(7, 8) model achieves the lowest NOE value combined with a maximum error that is second best [the NLR(6) maximum error is somewhat smaller]. Yet, it is characterized by the (by far) highest parametric complexity, which (due to the linearity of the method) is not reflected in the computational time (which, due to the nonlinear regression, is much higher for the NLR method). In terms of achieved accuracy, the DLR(7, 8) model is followed by the NLR(6) model, with the and LR(6) model being worst.

From these results it is evident that the NLR(6) model achieves drastically better simulation performance than its LR(6) counterpart, while both models are characterized by six operators. On the other hand, the DLR(7, 8) model achieves the lowest NOE criterion [less than half of that of the NLR(6)]. This is due to the versatility exhibited by the DLR models in accounting (through their significantly increased parametric complexity) for the extra dynamics that are, expectedly, present in an actual presliding friction experiment and which may not be accounted by the Maxwell Slip model structure. Nevertheless, the NLR(6) model (with estimated parameters presented in Table 3) strikes a very good balance between accuracy and parametric complexity. Its required computational time is clearly high, but this is not necessarily a concern as long as the model is estimated off line.
6 Conclusions

The problem of presliding friction dynamics identification based upon the Maxwell Slip model structure has been addressed. This structure is closely related to models with linear-stop operators, and is capable of accounting for the hysteresis with nonlocal memory phenomena that characterize the presliding friction dynamics. The model structure’s basic properties have been examined, based upon which a priori identifiability has been established. The role of initial conditions on identification has been also investigated, and the necessary and sufficient conditions for a posteriori identifiability have been derived. Using these results, guidelines for excitation signal design have been also formulated.

Building upon these results, two new methods, referred to as Dynamic Linear Regression (DLR) and NonLinear Regression (NLR), for presliding friction identification have been postulated. Their effectiveness has been assessed via Monte Carlo experiments with a Maxwell Slip model (Data Set A), as well as via experimental signals obtained from a physical system operating within the presliding friction regime (Data Set B). Comparisons with the conventional Linear Regression (LR) method that uses threshold preassignment have been also made. These results have indicated that:

(i) The LR, DLR, and NLR methods are capable of modelling the presliding friction dynamics, but with different degrees of accuracy.

(ii) The conventional LR method achieves partial model estimation and leads to the highest errors and asymptotically biased parameter estimates.

(iii) The DLR method also achieves partial model estimation. Yet, its structure allows for the representation of additional dynamics, and, also, for the alleviation (to a certain degree) of the effects of false threshold preassignment. The DLR generally achieves the lowest error, but, at the same time, exhibits the highest parametric complexity.

(iv) The NLR method achieves complete model identification via nonlinear regression. It attains excellent parameter accuracy, and strikes a very good balance between overall accuracy and parametric complexity.

It should be finally noted that most of the results of this study pertain to the Maxwell Slip model structure without being necessarily limited to the presliding friction case. As such, they may be used
for the identification of a number of systems or phenomena that may be adequately described by the Maxwell Slip model structure (for instance magnetic materials and piezo-actuators which also exhibit hysteresis with nonlocal memory characteristics).

Acknowledgement

The authors wish to acknowledge the financial support of this study by the VolkswagenStiftung (Grant no 1/76938). Special thanks are due to the Katholieke Universiteit Leuven (KUL) research team, led by Professors H. Van Brussel and F. Al-Bender, for providing the experimental friction data, guidance, and motivation. Also to all project partners for useful discussions and fruitful collaboration, as well as to two anonymous referees whose constructive comments and criticism helped in significantly improving the manuscript.
References


Appendix A: Proof of Proposition 3.1

Consider two Maxwell Slip models, subject to assumptions $A_1 - A_2$, with $M$ operators and parameters $k, d$ and $\bar{k}, \bar{d}$, respectively. Both are excited by a common $x(t)$ and provide identical responses, thus Eq.(6):

$$F_M(t) = \bar{F}_M(t) \iff \sum_{i=1}^{M} k_i \cdot \delta_i(t) = \sum_{i=1}^{M} \bar{k}_i \cdot \bar{\delta}_i(t), \; \forall t = 1, 2, \ldots$$ (26)

Without loss of generality, let the threshold ordering be (also see assumption $A_2$):

$$\bar{\Delta}_1 < \Delta_1 < \bar{\Delta}_2 < \Delta_2 < \ldots < \bar{\Delta}_M < \Delta_M$$ (27)

Also let the excitation $x(t)$ be ramped up to a maximum $x_{max} = x(t_{max})$, such that $\delta_i(t_{max}) = \Delta_i$ and $\bar{\delta}_i(t_{max}) = \bar{\Delta}_i$, $\forall i = 1, \ldots, M$ (Properties 2.1 and 2.2). This eliminates the initial state effects (Property 2.7). For $t > t_{max}$, let $x(t)$ decrease such that:

$$C_1 \; -2\bar{\Delta}_1 < x(t_{max} + 1) - x_{max} < 0$$

$$C_2 \; \forall j = 2, \ldots, M, \; \exists t_j > t_{max} + 1 : -2\Delta_j < x(t_j + 1) - x_{max} < x(t_j) - x_{max} \leq -2\Delta_{j-1}$$

Since $\delta_i(t_{max}) = \Delta_i$ and $\bar{\delta}_i(t_{max}) = \bar{\Delta}_i$, $\forall i = 1, \ldots, M$, Eq.(26) at time $t_{max}$ yields:

$$\sum_{i=1}^{M} k_i \cdot \Delta_i = \sum_{i=1}^{M} \bar{k}_i \cdot \bar{\Delta}_i$$ (28)

Adding Eqs.(26) and Eq.(28) yields:

$$\sum_{i=1}^{M} k_i \cdot [\delta_i(t) + \Delta_i] = \sum_{i=1}^{M} \bar{k}_i \cdot [\bar{\delta}_i(t) + \bar{\Delta}_i], \; \forall t$$ (29)

Since $\delta_i(t_{max}) = \Delta_i$, $\bar{\delta}_i(t_{max}) = \bar{\Delta}_i$ $\forall i = 1, \ldots, M$ and $-2\bar{\Delta}_1 < x(t_{max} + 1) - x_{max} < 0$ (C1), all state variables cross their positive stick border simultaneously at time $t_{max} + 1$ (Properties 2.3 $iii$ and 2.4; notice that $\bar{\Delta}_1$ is the smallest threshold).

As long as a state variable remains sticking for $t > t_{max}$, its evolution is given by Eq.(9) with $x(t_0) = x_{max}$ and $\delta_i(t_0) = \Delta_i$, that is:

$$\delta_i(t + 1) = x(t + 1) - x_{max} + \Delta_i$$ (30)

Substituting this into Eq.(29) for $t = t_{max} + 1$ and also using Eq.(28) yields:

$$\sum_{i=1}^{M} [k_i - \bar{k}_i] \cdot [x(t_{max} + 1) - x_{max}] = -2 \cdot \sum_{i=1}^{M} [k_i \cdot \Delta_i - \bar{k}_i \cdot \bar{\Delta}_i] = 0 \iff \sum_{i=1}^{M} [k_i - \bar{k}_i] = 0$$ (31)
Since \( x(t) \) decreases according to \( C2 \), then [due to Eq.(27)] at time \( t_j \) \( \forall j = 2, \ldots, M \) all state variables with thresholds lower than or equal to \( \Delta_{j-1} \) have already crossed their negative slip border (Property 2.6). Moreover, since \( x(t) \) decreases for \( t > t_{\text{max}} \), all state variables that are in negative slip at time \( t_j \) remain slipping for \( t > t_j \) [Property 2.3(i)]. Thus \( \forall j = 2, \ldots, M \):

\[
\delta_r(t) = -\Delta_r \quad \text{and} \quad \bar{\delta}_r(t) = -\bar{\Delta}_r, \quad \forall t \geq t_j \quad \text{and} \quad \forall r = 1, \ldots, j-1 \quad (32)
\]

Next consider \( j = M \). Conditions \( C1 \) and \( C2 \) imply that \( |\delta_M(t)| < \Delta_M, \forall t \in (t_{\text{max}}, t_M + 1] \) (the condition for crossing its negative slip border has not been fulfilled yet). It is not, however, certain whether or not the operator with state \( \bar{\delta}_M(t) \) remains sticking within this time interval. Assume, for the moment, that \( \bar{\delta}_M(t) = -\bar{\Delta}_M \) for \( t \in [t_M, t_M + 1] \). Eq.(29) for \( t = t_M \) and \( t = t_M + 1 \) [following Eq.(32)] gives, in both cases, the unacceptable solution \( k_M = 0 \) (Assumption A1). Therefore both \( \delta_M(t) \) and \( \bar{\delta}_M(t) \) remain sticking \( \forall t \in (t_{\text{max}}, t_M + 1] \), and their evolutions for this time interval are provided by Eq.(30). Setting \( t = t_M \) and \( t = t_M + 1 \) into Eq.(29) [while taking Eqs.(32) into account] and subtracting the two resulting equations yields:

\[
[k_M - \bar{k}_M] \cdot [x(t_M) - x(t_M + 1)] = 0 \iff k_M = \bar{k}_M \quad (33)
\]

while Eq.(29) for \( t = t_M \) gives \( \Delta_M = \bar{\Delta}_M \).

The iteration of this procedure for \( j = M - 1, \ldots, j = 2 \), each time using the results of the previous iteration, yields:

\[
k_j = \bar{k}_j \quad \text{and} \quad \Delta_j = \bar{\Delta}_j, \quad \forall j = 2, \ldots, M \quad (34)
\]

This, coupled with Eqs.(28) and (31), finally yields: \( k_1 = \bar{k}_1 \) and \( \Delta_1 = \bar{\Delta}_1 \). \( \square \)

8 Appendix B: Proof of Proposition 3.2

For the proof of this proposition it is sufficient to provide a case of a Maxwell Slip model that is not identifiable with respect to the initial state vector. This is done by demonstrating that a model features the exact same response when subject to two different initial state vectors \( \delta_1 \) and \( \bar{\delta}_1 \) (the excitation \( x(t) \) being the same).

Let a Maxwell Slip model with \( M = 2 \) operators and \( d = [\Delta_1 \Delta_2]^T, k = [k_1 \, k_2]^T \) such that \( k_1 = k_2 \). Let \( F_2(t) \) and \( \bar{F}_2(t) \) designate the responses due to the same \( x(t) \), but under the initial state vector
\[ \delta_1 = [\delta_1(1) \ \delta_2(1)]^T \] and \[ \tilde{\delta}_1 = [\tilde{\delta}_1(1) \ \tilde{\delta}_2(1)]^T, \] respectively. The \( \delta_i(t) \) and \( \tilde{\delta}_i(t) \) \( \forall i = 1, 2 \) evolutions start within their respective stick regimes (or else the initial state effects cannot be observed; Property 2.7).

Let \( |\delta_i(t)| < \Delta_i \) and \( |\tilde{\delta}_i(t)| < \Delta_i \), \( \forall t \in [1, t_1] \) and \( \forall i = 1, 2 \). It is easy to verify that the responses \( F_2(t) \) and \( \bar{F}_2(t) \) will coincide \( \forall t \in [1, t_1] \) iff [using Eq.(9)] \( \delta_1(1) + \delta_2(1) = \tilde{\delta}_1(1) + \tilde{\delta}_2(1) \), that is:

\[
F_2(t) = \bar{F}_2(t) \quad \forall t \in [1, t_1] \iff \delta_1(1) + \delta_2(1) = \tilde{\delta}_1(1) + \tilde{\delta}_2(1) \quad (35)
\]

Assume that the initial states satisfy the above, and also the following two conditions:

\[
\Delta_1 - \delta_1(1) = \Delta_2 - \tilde{\delta}_2(1) < \Delta_1 - \tilde{\delta}_1(1) = \Delta_2 - \delta_2(1) \quad (36)
\]

\[
\Delta_1 - \delta_1(1) \leq x(t_1 + 1) - x(1) < \Delta_1 - \tilde{\delta}_1(1) \quad (37)
\]

Then, due to Property 2.5, both \( \delta_1(t) \) and \( \tilde{\delta}_2(t) \) cross their positive slip border at \( t_1 + 1 \), while \( \delta_1(t) \) and \( \tilde{\delta}_2(t) \) remain sticking. Employing Eq.(9) again with \( \delta_1(t_1 + 1) = \Delta_1 \) and \( \tilde{\delta}_2(t_1 + 1) = \Delta_2 \), it is easy to verify that \( F_2(t_1 + 1) = \bar{F}_2(t_1 + 1) \). This indicates that the model provides identical response within the time interval \([1, t_1 + 1]\) for the two sets of initial conditions. Also notice that the responses for \( t > t_1 + 1 \) contain no further effects of the initial conditions (positive slip border crossing \( t_1 + 1 \) - Property 2.7).

Hence \( F_2(t) = \bar{F}_2(t) \) \( \forall t \) and the initial state vectors are not identifiable. \( \square \)

9 Appendix C: Proof of Proposition 3.4

Sufficiency: Similar with this of Proposition 3.1 [note that \( \delta_i(t_{cr}) = \Delta_i, \ \forall i = 1, \ldots, M \)].

Necessity of C1: If C1 is not valid, then at \( t_{cr} + 1 \) at least one \( \delta_r(t) \), with \( r \in [1, M] \), changes slip direction [Property 2.3 (ii)]. Hence, the a posteriori identifiability is lost, since any \( \delta_r(t) \), with arbitrary small \( \Delta_r \), changes slip direction at \( t_{cr} + 1 \).

Necessity of C2: The necessity of condition C2 is proved via counterexamples. Consider a Maxwell Slip model subject to assumptions A1 – A2 with M operators and parameter vectors \( k \) and \( d \). Let its response \( F_M(t) \) be due to an excitation \( x(t) \) that satisfies conditions C0 and C1 only. Since C2 is not satisfied, then as \( x(t) [t > t_{cr}] \) decreases, there is at least one \( j \in [2, M] \) such that \( \delta_j(t) \) and \( \delta_j(t) \) cross their negative slip border at the same time \( t_j > t_{cr} + 1 \) (case A) or at successive times \( t_j > t_{cr} + 1 \) and \( t_j + 1 \) (case B) [see Section 3.2.2 right after Proposition 3.4].
Case A: Consider that $\delta_{j-1}(t)$ and $\delta_j(t)$ cross their negative slip borders at the same time $t_j$ [$\Delta_{j-1}$ and $\Delta_j$ subject to Eq.(14)]. Assume another Maxwell Slip model, subject to Assumptions A1 – A2, with $M$ operators and parameter vectors $\bar{k}$ and $\bar{d}$. Let it be excited by the same $x(t)$, providing response $\bar{F}_M(t)$. Assume that only conditions $C0$ and $C1$ are valid for this model, as well. This model’s parameters are selected such that $k_i = \bar{k}_i$ and $\Delta_i = \bar{\Delta}_i$, $\forall i \in [1, j - 2] \cup \{j + 1, M\}$ and $\bar{k}_{j-1}$, $\bar{k}_j$, $\bar{\Delta}_{j-1}$ and $\bar{\Delta}_j$ subject to:

\[
k_{j-1} + k_j = \bar{k}_{j-1} + \bar{k}_j \tag{38}
\]
\[
k_{j-1} \cdot \Delta_{j-1} + k_j \cdot \Delta_j = \bar{k}_{j-1} \cdot \bar{\Delta}_{j-1} + \bar{k}_j \cdot \bar{\Delta}_j \tag{39}
\]
\[
x(t_j) - x(t_{cr}) \leq -2 \cdot \Delta_j \leq -2 \cdot \bar{\Delta}_j < -2 \cdot \Delta_{j-1} \leq -2 \cdot \bar{\Delta}_{j-1} < x(t_j - 1) - x(t_{cr}) \tag{40}
\]

Now subtract the responses [Eq.(6)] of the two models, keeping in mind that some of their parameters are identical (as above). This yields:

\[
F_M(t) - \bar{F}_M(t) = \sum_{r=j-1}^{j} \left[ k_r \cdot \delta_r(t) - \bar{k}_r \cdot \bar{\delta}_r(t) \right], \quad \forall t \geq t_{cr} \tag{41}
\]

Since condition $C1$ is valid for both models, then all state variables cross their positive stick border at time $t_{cr} + 1$. Owing to Eq.(40) and Property 2.5, $\delta_{j-1}(t)$ and $\delta_j(t)$, as well as $\bar{\delta}_{j-1}(t)$ and $\bar{\delta}_j(t)$, cross their negative slip borders at time $t_j$ (condition $C2$ is not satisfied). Thus, these state variables remain sticking for the $[t_{cr} + 1, t_j - 1]$ time interval, and, therefore, their corresponding evolutions, within this interval, are given by Eq.(9) (with $x(t_0) = x(t_{cr})$, $\delta_r(t_0) = \Delta_r$ and $\bar{\delta}_r(t_0) = \bar{\Delta}_r$, $\forall r \in [j - 1, j]$). Due to this and Eqs.(38) - (39), Eq.(41) becomes:

\[
F_M(t) - \bar{F}_M(t) = \sum_{r=j-1}^{j} \left( k_r - \bar{k}_r \right) \cdot \left( x(t) - x(t_{cr}) \right) + \sum_{r=j-1}^{j} \left( k_r \cdot \Delta_r - \bar{k}_r \cdot \bar{\Delta}_r \right) = 0, \quad \forall t \in [t_{cr} + 1, t_j - 1] \tag{42}
\]

Since $x(t)$ decreases $\forall t > t_{cr}$, then according to Property 2.3(i), $\delta_{j-1}(t)$, $\delta_j(t)$, $\bar{\delta}_{j-1}(t)$ and $\bar{\delta}_j(t)$ remain within the negative slip regime $\forall t \geq t_j$. Thus Eq.(41), using Eq.(39), becomes:

\[
F_M(t) - \bar{F}_M(t) = - \sum_{r=j-1}^{j} \left( k_r \cdot \Delta_r - \bar{k}_r \cdot \bar{\Delta}_r \right) = 0, \quad \forall t \geq t_j \tag{43}
\]

Eqs.(42) - (43), combined with the fact that at $t_{cr}$ all state variables are in positive slip, imply that $F_M(t)$ and $\bar{F}_M(t)$ coincide $\forall t \geq t_{cr}$. Thus a posteriori identifiability is not possible.

Case B: Now assume that $\delta_{j-1}(t)$ and $\delta_j(t)$ cross their negative slip borders at successive time instants $t_j$ and $t_j + 1$, respectively [$\Delta_{j-1}$ and $\Delta_j$ subject to Eq.(15)]. Assume a second Maxwell Slip model, similar
to the previous but with $\tilde{k}_{j-1}$, $\bar{k}_j$, $\bar{\Delta}_{j-1}$ and $\bar{\Delta}_j$ subject to Eqs.(38)-(39) and:

$$
[k_j - \bar{k}_j] \cdot [x(t_j) - x(t_{cr})] = -2 \cdot [k_j \cdot \Delta_j - \bar{k}_j \cdot \bar{\Delta}_j] \tag{44}
$$

$$
x(t_j) - x(t_{cr}) \leq -2 \cdot \Delta_{j-1} \leq -2 \cdot \bar{\Delta}_{j-1} < x(t_j - 1) - x(t_{cr}) \tag{45}
$$

$$
x(t_j + 1) - x(t_{cr}) \leq -2 \cdot \Delta_j \leq -2 \cdot \bar{\Delta}_j < x(t_j) - x(t_{cr}) \tag{46}
$$

If a similar to the previous algebraic procedure is applied, keeping in mind that here (since $C2$ is not satisfied) both $\delta_{j-1}(t)$ and $\bar{\delta}_{j-1}(t)$ cross their negative slip borders at time $t_j$ [Eq.(45) and Property 2.5] and that both $\delta_j(t)$ and $\bar{\delta}_j(t)$ cross their negative slip borders at $t_j + 1$ [Eq.(46) and Property 2.5], it is concluded that the two models provide identical responses. Thus a posteriori identifiability is, once again, impossible.
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Model based friction force simulation (slider) contrasted to the actual friction force (−) and the corresponding error for each estimated model (Data Set B; part of the validation set).
Presliding Friction Identification Based Upon the Maxwell Slip Model Structure

Table 1: Identification results by the LR and NLR methods for Data Set A: No noise (30 Monte Carlo runs; point estimates plus/minus two standard deviations).

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<th>Type III</th>
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<td>$k_1$</td>
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<td>5.8 ± 1.4</td>
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<td>4.0 ± 1.9 x 10^{-3}</td>
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<tr>
<td>$k_2$</td>
<td>2.0</td>
<td>2.5 ± 2.1</td>
<td>2.0 ± 7.3 x 10^{-3}</td>
<td>2.0 ± 1.0 x 10^{-2}</td>
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<tr>
<td>$k_3$</td>
<td>1.0</td>
<td>0.2 ± 2.0</td>
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<td>1.0 ± 9.4 x 10^{-3}</td>
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<td>$\Delta_1$</td>
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<tr>
<td>$\Delta_2$</td>
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<tr>
<td>$\Delta_3$</td>
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<td>$\delta_1(1)$</td>
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<td>−</td>
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<tr>
<td>$\delta_2(1)$</td>
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<td>(0.3 ± 1.1) x 10^{-3}</td>
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Table 2: Characteristics of the estimated models (Data Set B: validation set).

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<th>NLR(6)</th>
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<td>Maximum error (Nt)</td>
<td>1.2</td>
<td>1.0</td>
<td>0.9</td>
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<td>Parametric complexity</td>
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<td>63</td>
<td>12</td>
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<tr>
<td>Computational time (min)</td>
<td>$1.8 \times 10^{-3}$</td>
<td>$3.4 \times 10^{-3}$</td>
<td>18.8</td>
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*aSimulation starting at $t = 18, 252$.

Table 3: The NLR(6) model estimated stiffness and threshold parameters (Data Set B).

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<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Threshold $\Delta_i$ (µm)</td>
<td>$5.7 \times 10^{-3}$</td>
<td>$1.1 \times 10^{-1}$</td>
<td>$4.2 \times 10^{-4}$</td>
<td>1.5</td>
<td>4.1</td>
<td>7.8</td>
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<tr>
<td>Stiffness $k_i$ (Nt/µm)</td>
<td>26</td>
<td>1.3</td>
<td>$4.1 \times 10^{-1}$</td>
<td>$4.3 \times 10^{-1}$</td>
<td>$2.2 \times 10^{-1}$</td>
<td>$5.9 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
Figure 1: The Maxwell Slip model structure.

Figure 2: The behavior of a single elasto-slide operator: (a) The exerted displacement; (b) resulting $\delta_1(t)$; (c) $x(t)$ versus $\delta_1(t)$ [$\Delta_1 = 0.5$]

Figure 3: Maxwell Slip model force $F_M(t)$ versus exerted displacement $x(t)$ [model with 10 operators; exerted displacement: $x(t) = \sin(0.2\pi t) \cdot \sin(5\pi t)$; sampling frequency: $f_s = 150 \text{ Hz}$].
Figure 4: Schematic representation of the Maxwell Slip model basic properties I: (a) Exerted displacement \([\text{local maximum at } t_0]\), (b)-(c) spring deformation evolutions \(\delta_1(t)\) and \(\delta_2(t)\) of operators with \(\Delta_1 = 0.2\) and \(\Delta_2 = 0.7\), respectively \([\text{Positive slip border crossing for } \delta_1(t) \text{ at } t'; \text{ Slip direction change and positive stick border crossing at } t_0 + 1 \text{ for } \delta_1(t) \text{ and } \delta_2(t), \text{ respectively}]\).

Figure 5: Schematic representation of the Maxwell Slip model basic properties II: (a) Exerted displacement \([\text{local maximum at } t_{\text{max}}]\), (b) spring deformation evolutions \(\delta_1(t)\) and \(\delta_2(t)\) of operators with \(\Delta_1 = 0.15\) and \(\Delta_2 = 0.5\), respectively \([\text{simultaneous positive stick border crossing at time } t_{\text{max}} + 1; \delta_2(t) \text{ crosses its positive and negative slip border at } t_2 \text{ and } t', \text{ respectively}]\), (c) spring deformation evolution \(\delta_2(t)\) starting from different initial states \(\delta_2(1)\) and \(\bar{\delta}_2(1)\) \([\text{initial state effect elimination}]\).
Presliding Friction Identification Based Upon the Maxwell Slip Model Structure

Figure 6: Contour plot of the cost function $J$ versus a single stiffness and threshold parameter set [NLR(1) model].

Figure 7: NOE criterion versus model order (Data Set A; 5% noise).

Figure 8: Identification results for Data Set A in the 5% noise case: (a) NOE criterion; (b) stiffness estimates; (c) threshold estimates [30 Monte Carlo runs; the boxes indicate point estimates plus/minus two standard deviations; the dashed horizontal lines indicate actual parameter values; LR(3) model (−□−) and NLR(3) model (−)].
Figure 9: NOE criterion versus model order: (a) Estimation set; (b) validation set (Data Set B).

Figure 10: Model based friction force simulation (·) contrasted to the actual friction force (−) and the corresponding error for each estimated model (Data Set B; part of the validation set).